**MATH2050B 1920 Quiz 2**

TA's solutions[1] to selected problems

Unless explicitly stated otherwise, $A \subset \mathbb{R}$, $l \in \mathbb{R}$, $x_0 \in A^c$(cluster pt w.r.t. $A$).

**Q1.** Given the definition and its negation for each of the following:

(i) $(x_n)$ is Cauchy (a Cauchy sequence).

(ii) $x_0$ is a cluster point w.r.t. a set $A$ of real numbers.

(iii) $f(x)$ converges to a real number $l$ as $x \to x_0$.

**Solution.**

(i) **Definition:** For any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n > N$, $|x_n - x_m| < \epsilon$.

**Negation:** There is some $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there is $m, n > N$ with $|x_n - x_m| \geq \epsilon$.

(ii) **Definition:** For any $\epsilon > 0$, there is some $a \in A$, $a \neq x_0$ and $|a - x_0| < \epsilon$.

**Negation:** There is some $\epsilon > 0$ such that $|x - a| \geq \epsilon$ for all $a \in A$.

(iii) **Definition:** For any $\epsilon > 0$, there is $\delta > 0$ so that for all $y$ with $0 < |y - x| < \delta$, one has $|f(x) - f(y)| < \epsilon$.

**Negation:** There is $\epsilon > 0$ such that for all $\delta > 0$, there is $y$ with $0 < |y - x| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

**Q2.** State (without proof)

(i) Monotone Convergence Theorem and Monotone Subsequence Theorem (on existence of some subsequence)

(ii) Order-preserving Theorem and Squeeze Theorem for limits of functions

**Solution.**

(i) (Monotone Convergence Theorem) Any bounded monotone sequence of real numbers is convergent.

(Monotone Subsequence Theorem) Any sequence of real numbers has a monotone subsequence.

(ii) (Order-preserving Theorem) Let $A \subset \mathbb{R}$ be non-empty, $x_0 \in A^c$, $f, g : A \to \mathbb{R}$ be functions. Suppose that $f(x) \leq g(x)$ for all $x \in A \backslash \{x_0\}$ and suppose that $\lim_{x \to x_0} f(x)$, $\lim_{x \to x_0} g(x)$ exist. Then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x_0)$.

(Squeeze Theorem) Let $A \subset \mathbb{R}$ be non-empty, $x_0 \in A^c$, $f, g, h : A \to \mathbb{R}$ be functions. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in A \backslash \{x_0\}$ and suppose that $\lim_{x \to x_0} f(x)$, $\lim_{x \to x_0} h(x)$ exist and are equal. Then $\lim_{x \to x_0} g(x)$ exists and

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x).$$

[1]please kindly send an email to `ncl1iu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

**Q3.** State and prove that Bolzano-Weierstrass Theorem (allow to use **Q2**).

**Solution.** (Bolzano-Weierstrass Theorem) A bounded sequence of real numbers has a convergent subsequence.

Let $(x_n)$ be a bounded sequence of real numbers. By Monotone Subsequence Theorem, $(x_n)$ has a monotone subsequence $(x_{n_k})$. Because $(x_{n_k})$ is bounded, so it is convergent by Monotone Convergence Theorem.

**Q4.** Let $x_1 = 4$ and $x_{n+1} = \frac{1}{3}(x_n + 5)$. Show that $(x_n)$ is convergent and find its value.

**Solution.** We claim that $(x_n)$ is decreasing. First, $x_2 = 3 \leq x_1$. Suppose that $x_k \leq x_{k-1}$, then

$$x_k + 5 \leq x_{k-1} + 5,$$

so

$$\frac{1}{3}(x_k + 5) \leq \frac{1}{3}(x_{k-1} + 5).$$

Hence $x_{k+1} \leq x_k$. By MI $(x_n)$ is decreasing.

Next we claim that $(x_n)$ is bounded below. First, $x_1 \geq 0$ and if $x_k \geq 0$, then $x_{k+1} = \frac{1}{3}(x_k + 5) \geq 0$. By MI $x_n \geq 0$ for all $n$.

By MCT $x_n$ is convergent, say $L = \lim_n x_n$. Because $x_{n+1} = \frac{1}{3}(x_n + 5)$, so

$$L = \frac{1}{3}(L + 5).$$

Hence $L = \frac{5}{2}$.

**Q5.** Use the definition in $\epsilon$-$\delta$ terminology, show that

(i) $\lim_{x\to 3} \frac{x^2+1}{x-2} = 10$.

(ii) If $\lim_{x\to x_0} f_i(x) = l_i$ $(i = 1, 2)$ then $\lim_{x\to x_0}(f_1(x)f_2(x)) = l_1 l_2$.

**Solution.** (*i*) First note that

$$|\frac{x^2+1}{x-2} - 10| = |\frac{x^2 - 10x + 21}{x-2}| = |\frac{(x-7)(x-3)}{x-2}|.$$

Second note that if $0 < |x - 3| < \frac{1}{2}$, then

$$\frac{5}{2} < x < \frac{7}{2},$$

so

$$\frac{1}{2} < x - 2 < \frac{3}{2}.$$

Thus $|x - 2| > \frac{1}{2}$ for all $x$ with $0 < |x - 3| < \frac{1}{2}$. On the other hand,

$$-\frac{9}{2} < x - 7 < -\frac{7}{2}.$$

Thus $|x - 7| < \frac{9}{2}$ for all $x$ with $0 < |x - 3| < \frac{1}{2}$.

Now we show that $\lim_{x\to 3} \frac{x^2+1}{x-2} = 10$. Let $\epsilon > 0$. Set $\delta = \min\{\epsilon\frac{1}{9}, \frac{1}{2}\} > 0$, then for all $x$ with $0 < |x-3| < \delta$, we have

$$\begin{aligned} |\frac{x^2+1}{x-2} - 10| &= \frac{|x-7| \cdot |x-3|}{|x-2|} \\ &< \frac{9}{2}\frac{2}{1}\epsilon\frac{1}{9} \\ &= \epsilon. \end{aligned}$$

Hence $\lim_{x\to 3} \frac{x^2+1}{x-2} = 10$.

$(ii)$ First note that

$$|f_1(x)f_2(x) - l_1l_2| = |f_1(x)f_2(x) - f_1(x)l_2 + f_1(x)l_2 - l_1l_2| \le |f_1(x)| \cdot |f_2(x) - l_2| + |l_2| \cdot |f_1(x) - l_1|.$$

Second, we claim that there is $\delta' > 0$ and $M > 0$ so that for any $x$ with $0 < |x - x_0| < \delta'$, we have $|f_1(x)| \le M$ .

Let $\epsilon' = 1$. Then there is $\delta'$ so that $|f_1(x) - l_1| < 1$ for all $x$ with $0 < |x - x_0| < \delta'$. This implies that $-1 + l_1 < f_1(x) < 1 + l_1$ for all $x$ with $0 < |x - x_0| < \delta'$. Therefore there is a large $M > 0$ so that

$$-M < -1 + l_1 < f_1(x) < 1 + l_1 < M,$$

giving $|f_1(x)| < M$ for all $x$ with $0 < |x - x_0| < \delta'$.

Let $\epsilon > 0$. By assumption there is $\delta'' > 0$ so that for any $x$ with $0 < |x - x_0| < \delta''$, we have

$$|f_1(x) - l_1| < \epsilon\frac{1}{2(|l_2| + 1)}, \qquad |f_2(x) - l_2| < \epsilon\frac{1}{2M}.$$

Set $\delta = \min\{\delta', \delta''\} > 0$. For any $x$ with $0 < |x - x_0| < \delta$, we have

$$\begin{aligned} |f_1(x)f_2(x) - l_1l_2| &\le |f_1(x)| \cdot |f_2(x) - l_2| + |l_2| \cdot |f_1(x) - l_1| \\ &< M \cdot \epsilon\frac{1}{2M} + |l_2|\epsilon\frac{1}{2(|l_2| + 1)} \\ &\le \epsilon. \end{aligned}$$

Hence $\lim_{x\to x_0}(f_1(x)f_2(x)) = l_1l_2$.